Übung Formale Grundlagen

Relations, Functions

Ondřej Smetana

ondrej.smetana@uni-leipzig.de
ond.smetana@gmail.com

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Last Time

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Sets, Set Operations, Set Equivalences

- Union, Intersection, Complement, Difference
- Various laws, DeMorgan's Laws

Set Equivalences

What to remember?

1. Remember the names
2. Write down what equivalence/law you used

Recap: Set theoretic equalities

(1) **Idempotent Laws**:

a. $X \cup X = X$

b. $X \cap X = X$

(2) **Commutative Laws**:

a. $X \cup Y = Y \cup X$

b. $X \cap Y = Y \cap X$

(3) **Associative Laws**:

a. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

b. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

(4) **Identity Laws**:

a. $X \cup \varnothing = X$

b. $X \cap \varnothing = \varnothing$

c. $X \cup U = U$

d. $X \cap U = X$

(5) **Distributive Laws**:

a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(6) **Complement Laws**:

a. $A \cup A' = U$

b. $(A')' = A$

c. $A \cap A' = \varnothing$

d. $A - B = A \cap B'$

(7) **DeMorgan's Laws**:

a. $(A \cup B)' = A' \cap B'$

b. $(A \cap B)' = A' \cup B'$

(8) **Consistency Principle**:

a. $X \subseteq Y$ iff $X \cup Y = Y$

b. $X \subseteq Y$ iff $X \cap Y = X$

Example

Simplify the following expression.

$$((A \cup B') \cap (A \cup B)) - (A - B)$$

Relations

Definition (Tuple)

A **tuple** (Tupel) is an *ordered* collection of elements. Members of a tuple are enclosed in "⟨⟩".

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Definition (Tuple, v2)

A tuple of elements a and b is defined as

$$\langle a, b \rangle = \{\{a\}, \{a, b\}\}.$$

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Ordered

It is important to see $\langle a, b\rangle \neq \langle b, a\rangle$.

Cartesian Product

Definition (Cartesian Product)

Let A, B be sets. The **Cartesian product** (kartesisches Produkt) $A \times B$ is the set of all ordered pairs with the first element from A and the other from B.

$$A \times B = \{\langle a, b\rangle \,|a \in A, b \in B\}$$

Example

Consider the sets

- D(eterminers) = {'every', 'the'},
- N(ouns) = {'flower', 'day', 'child'}.

Then $D \times N$ is the set of all valid 2-word determiner phrases:
{⟨'every', 'flower'⟩ , ⟨'every', 'day'⟩ , ⟨'every', 'child'⟩ ,
⟨'the', 'flower'⟩ , ⟨'the', 'day'⟩ , ⟨'the', 'child'⟩}

Motivation

We want to use formal language to describe natural language (to capture meaning, ambiguity). Sets are enough to describe events with no objects (intransitive; *sleep, shout, sit, ...*).

What about events with two individuals? X *buys* Y, A *sees* B, F *reads* L, ...(transitive).

We are looking for a way to capture the *relation* between the two.

Definition (Relation)

Let X and Y be sets. The **relation** (Relation) R is a subset of $X \times Y$.
The smallest sets $A \subseteq X$ and $B \subseteq Y$ such that $R \subseteq A \times B$ are called *domain* (Definitionsbereich) and *range/codomain* (Zielmenge) of R, respectively.

Relation Examples

Relation Example 1

Example

Let us have sets A and B:

$$A = \{\text{🐱}, \text{🐶}, \text{🧍}, \text{🐴}\},$$
$$B = \{\text{🍗}, \text{🍩}, \text{🍖}, \text{🍕}\}$$

The Cartesian product $A \times B$:

$$\{\langle\text{🐱},\text{🍗}\rangle,\langle\text{🐱},\text{🍩}\rangle,\langle\text{🐱},\text{🍖}\rangle,\langle\text{🐱},\text{🍕}\rangle,\langle\text{🐶},\text{🍗}\rangle,\langle\text{🐶},\text{🍩}\rangle,\langle\text{🐶},\text{🍖}\rangle,\langle\text{🐶},\text{🍕}\rangle,$$
$$\langle\text{🧍},\text{🍗}\rangle,\langle\text{🧍},\text{🍩}\rangle,\langle\text{🧍},\text{🍖}\rangle,\langle\text{🧍},\text{🍕}\rangle,\langle\text{🐴},\text{🍗}\rangle,\langle\text{🐴},\text{🍩}\rangle,\langle\text{🐴},\text{🍖}\rangle,\langle\text{🐴},\text{🍕}\rangle\}$$

Let's look at the relation "eats" and "is eaten".

Relation Example 1

Example (Relation “eats”)



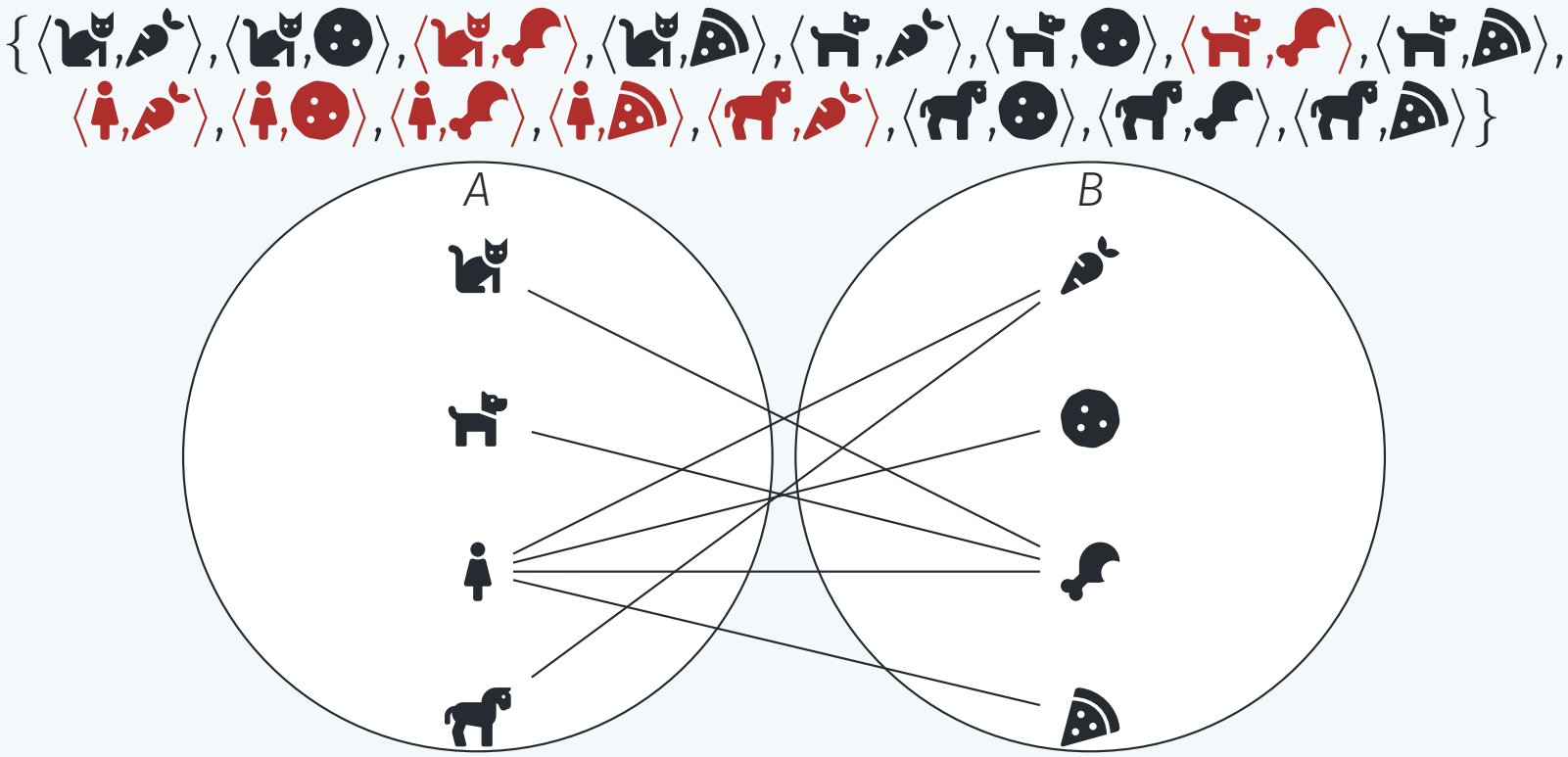
Relation Example 1

Example (Relation “eats”)



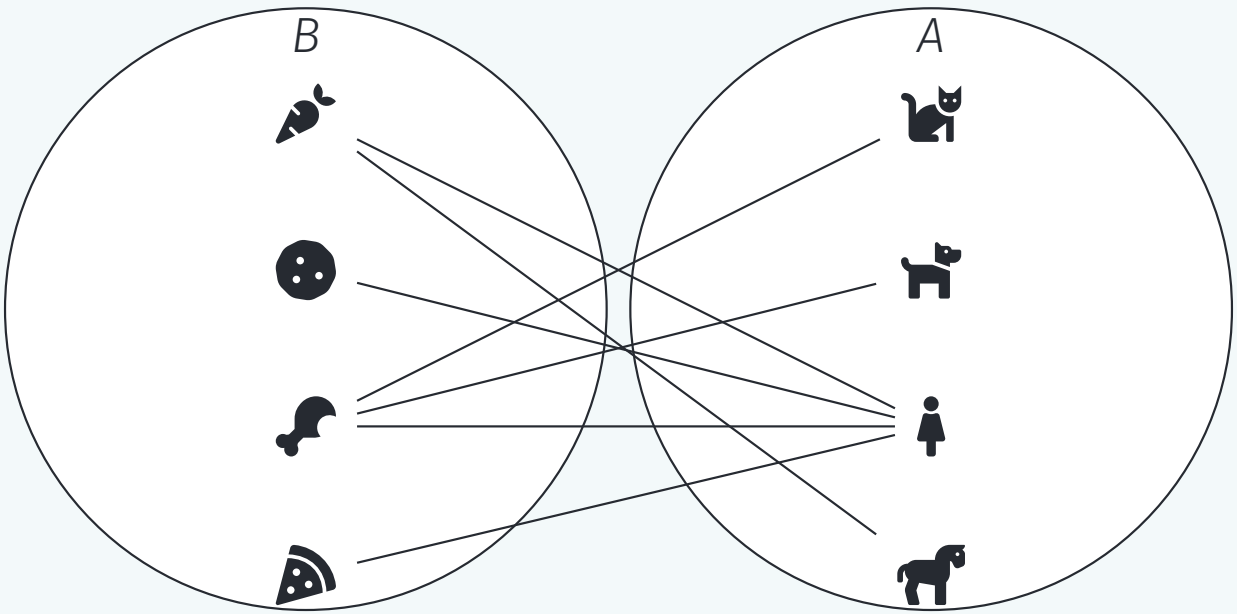
Relation Example 1

Example (Relation “eats”)



Relation Example 1

Example (Relation “is eaten” (inverse to “eats”))



Relation Example 1

Example (Complementary relation to the relation “eats”)

Relation “eats”

Complementary relation to “eats”



Relation Example 1

Example (Complementary relation to the relation “eats”)

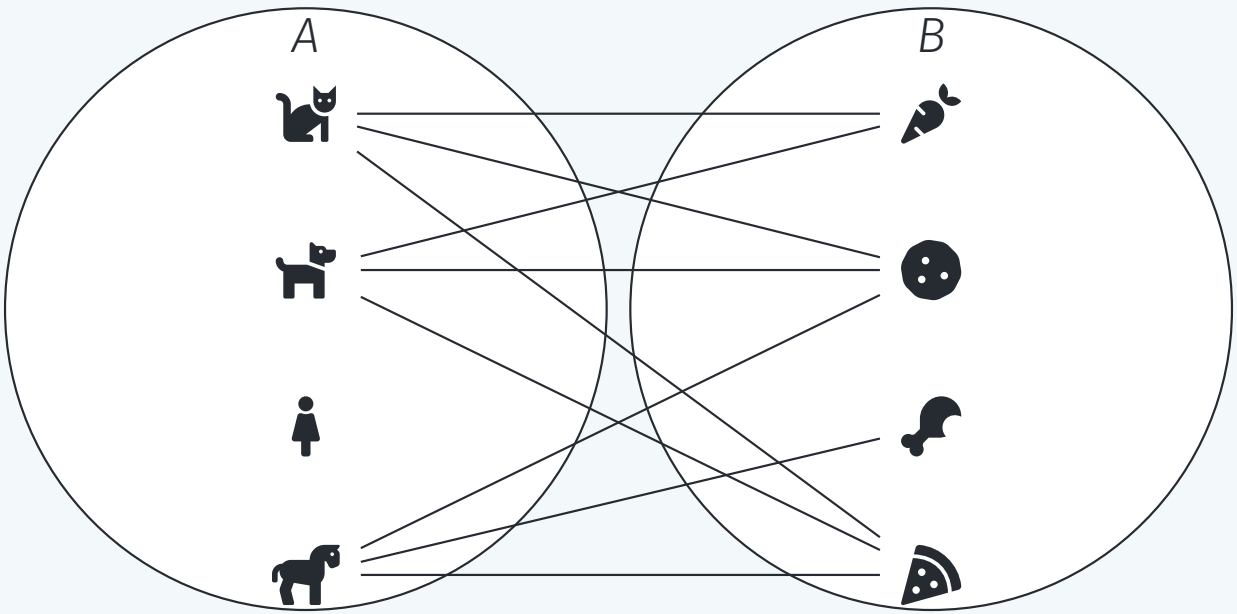
Relation “eats”

Complementary relation to “eats”



Relation Example 2

Example (Complementary relation to the relation “eats”)



Relation Example 2

Example

Let A be a set of consonants {[p], [b], [t], [d]}. The Cartesian product of $A \times A$ contains 16 pairs/tuples/elements. We are interested in the relation

"shares the exact same voicing".

We define $R = \{\langle x, y \rangle \,|x, y$ share the same voicing quality$\}$.

Relation Example 3

Example

Let F be the relation

"is a geographic sub-region of."

Let

- $G = \{\text{Germany, France}\}$,
- $L = \{\text{Saxony, Brandenburg, Thüringen, ...}\} = \{x | x \text{ is a German state}\}$.

Does order matter? Is $F \subseteq G \times L$, or $F \subseteq L \times G$?

Functions

Terminology

Let A and B be sets, R a relation (ie., $R \subseteq A \times B$). The relation R is a **total function** if

1. each element in A is only paired with one element in the range or R *(function)*
2. the domain of R is A *(total)*

Notation

Let R be a function ($R \subseteq A \times B$). We say that R is a function from A to B, $R : A \to B$.
For $a \in A$ and $b \in B$ such that $\langle a, b \rangle \in R$, we write $R(a) = b$ ($R : a \mapsto b$).

Terminology

Let us have a function $f : A \to B$.

- If f is not total, it is called a **partial** function.
- f is **1-to-1** (injective) if for every $a \in A$ there is only one $b \in B$ such that $f(a) = b$.
- f is **into** (B) if the range of f is a proper subset of B
- f is **onto** (B) (surjective) if the range of f is B.
- f is **bijective** if it is 1-to-1 and onto.

Function Example 1

Function Example 1

Example

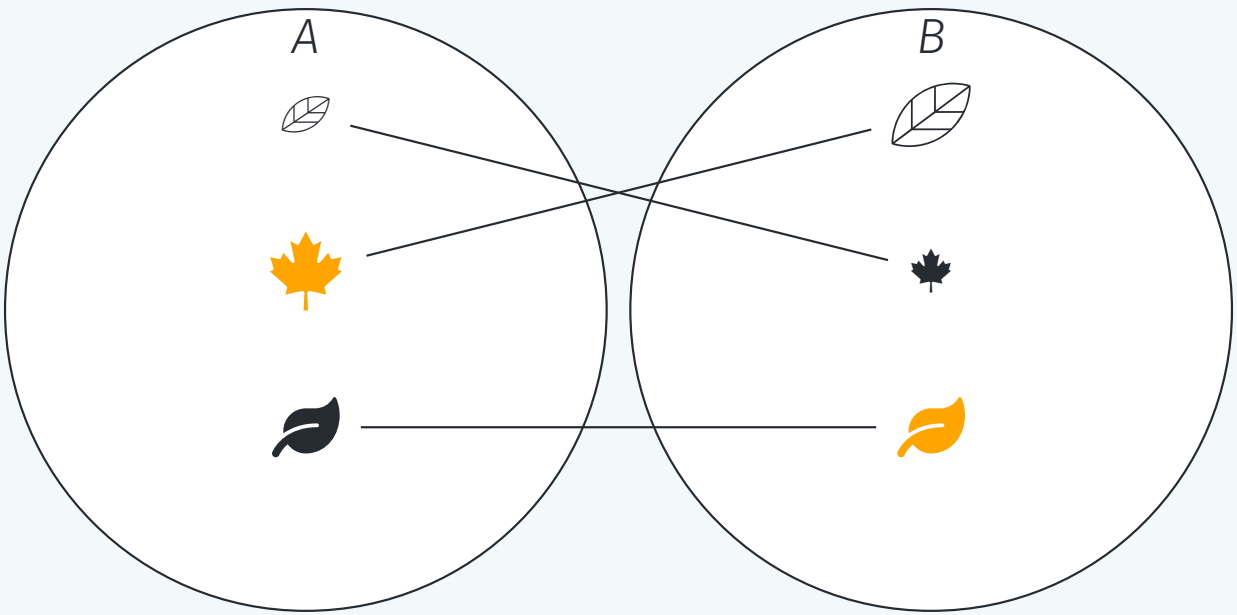
Let us have sets $A = \{$, , $\}$, $B = \{$, , $\}$. Draw diagrams for the following relations:

1. $R_1 \subseteq A \times B$: "is the same size as",
2. $R_2 \subseteq A \times B$: "has the same color as".

- Decide if R_1, R_2 are functions.
- Is R_1^{-1} or R_2^{-1} a function?
- Are *complementary* relations to R_1, R_2 functions?

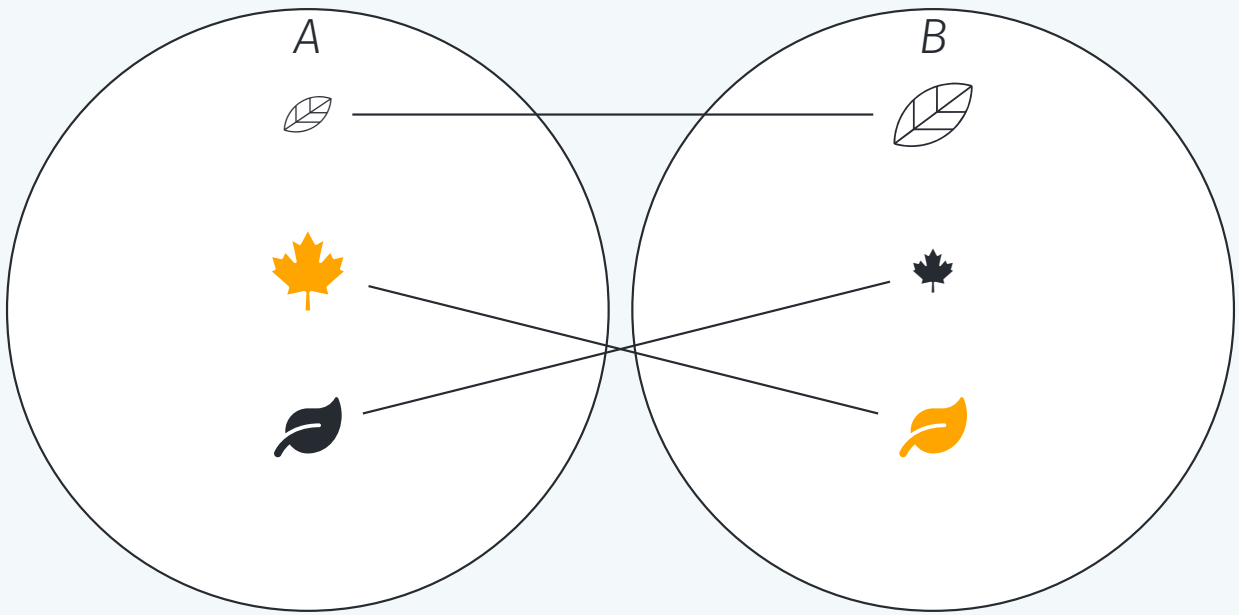
Function Example 1

Example ($R_1 \subseteq A \times B$: "is the same size as")



Function Example 1

Example ($R_2 \subseteq A \times B$: "has the same color as")



Excursion to Formal Semantics

Compositional Semantics

- Compositional semantics: meaning is built from smaller pieces, we wish to capture that formally
- λ-calculus
  - John reads a book.
  - reads(`John`, `book`)
  - reads(x, `book`)
  - λx.reads(x, `book`)
  - $\lambda y.\lambda x$.reads(x, y)

Function Example 2

Function Example 2

Favorite Animal

1. Work in groups
2. Ask your group mates about their favorite animal
3. Write down a function that assigns each individual their favorite animal
   - Use both predicate notation and a list of tuples
   - Draw a diagram
4. Write down the range and domain of your function
5. Decide if the function is 1-to-1, surjective, bijective

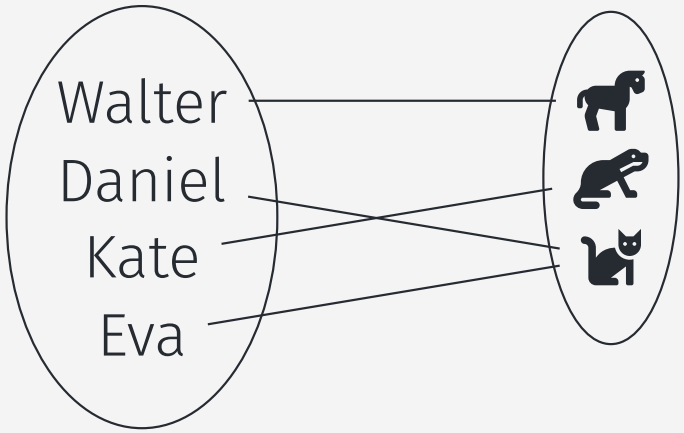
Function Example 2

Favorite Animal

- Domain: your group $A = \{\text{Walter, Daniel, Kate, Eva}\}$
- Range: the list of favorite stuff $B = \{$🐎, 🐊, 🐈$\}$
- f in predicate notation: $f = \{\, \langle x, y \rangle \mid$
  x is a group member and y is x's favorite animal $\}$
- f in list notation:
  $f = \{\, \langle \text{Walter}, $🐎$ \rangle, \langle \text{Daniel}, $🐈$ \rangle, \langle \text{Kate}, $🐊$ \rangle, \langle \text{Eva}, $🐈$ \rangle \}$

Function Example 2

Favorite Animal



Function Composition

Composition

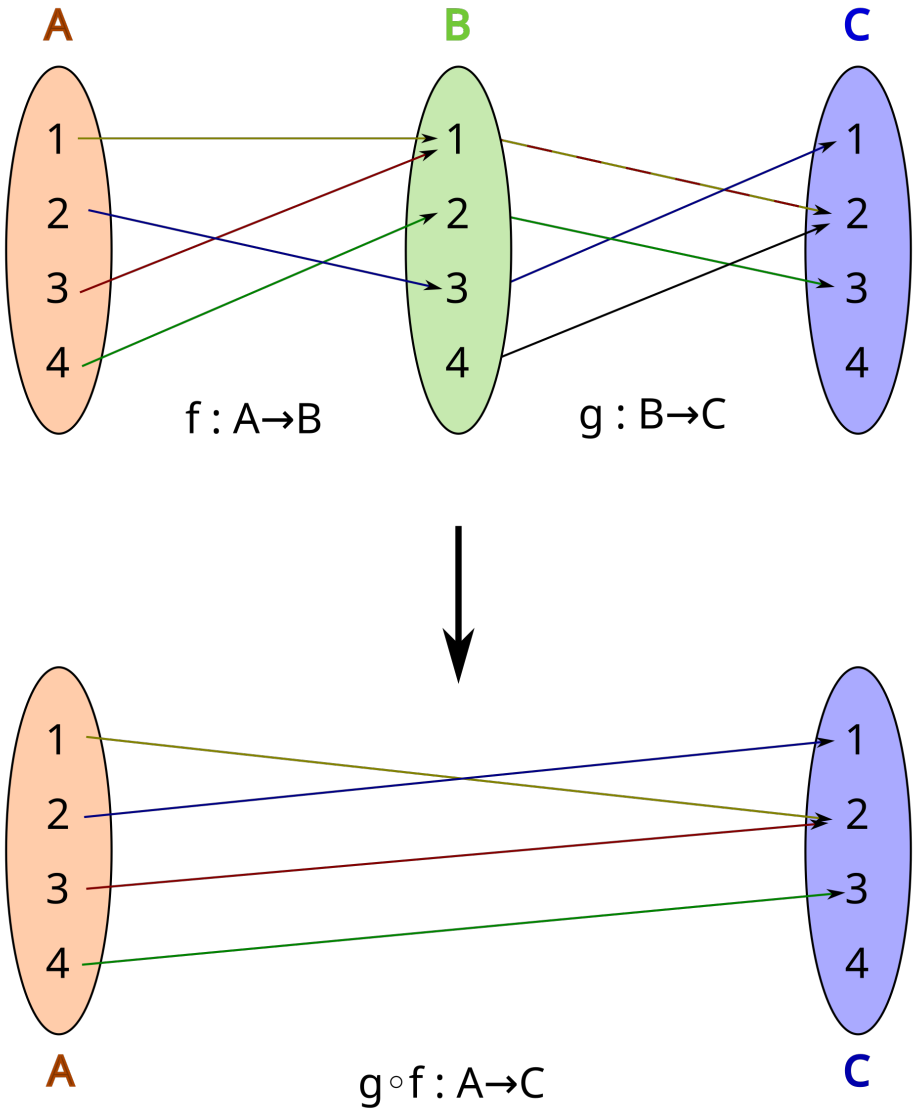
- Functions can be chained (composed)
- The output of one function is the input of the next one

Terminology and Notation

Let $f : A \to B$ and $g : B \to C$ be functions. The composite of f and g is the function $g \circ f : A \to B$. For $a \in A$,

$$(g \circ f)(a) = g(f(a))$$

Function Composition



Function Example 3

Function Example 3

Favorite Place

Let $C = \{$🌊, 🛋, 🌲$\}$. Let f be the function that assigns a person their favorite animal. Let g be the function that assigns a favorite place to an animal.
The function $g \circ f$ maps from the set of individuals to the set of places.

Function Example 3

Favorite Place

$$A = \{\text{Walter, Daniel, Kate, Eva}\}$$

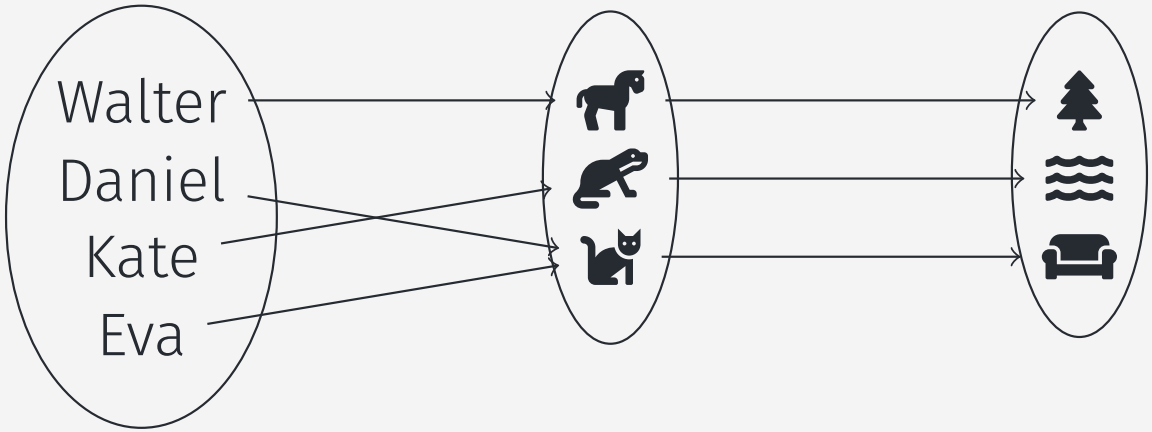
$$B = \{🐎, 🐊, 🐈\}$$

$$C = \{🌊, 🛋, 🌲\}$$

$$g = \{\langle 🐎, 🌲\rangle, \langle 🐊, 🌊\rangle, \langle 🐈, 🛋\rangle\}$$

Function Example 3

Favorite Animal



$$g\left(f\left(\text{Walter}\right)\right) = g\left(\text{🐴}\right) = \text{🌲}$$
$$g\left(f\left(\text{Daniel}\right)\right) = g\left(\text{🐈}\right) = \text{🛋}$$
$$g\left(f\left(\text{Kate}\right)\right) = g\left(\text{🐊}\right) = \text{🌊}$$
$$g\left(f\left(\text{Eva}\right)\right) = g\left(\text{🐈}\right) = \text{🛋}$$

Properties of Binary Relations

Notation and Terminology

A relation $R \subseteq A \times A$ is

- **reflexive** if for all $a \in A$: $\langle x, x \rangle \in R$
- **symmetric** if for all $a, b \in A$: if $\langle a, b \rangle \in R$, then $\langle b, a \rangle \in R$
- **antisymmetric** if for all $a, b \in A$: if $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$, then $a = b$
- **transitive** if for all $a, b, c \in A$: if $\langle a, b \rangle \in R$ and $\langle b, c \rangle \in R$, then $\langle a, c \rangle \in R$
- **connected** if for all $a, b \in A$, $a \neq b$: $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$

Binary Relation Example

Example

For each relation property (reflexivity, symmetry, antisymmetry, transitivity, connectedness), find a relation with such property.

1. "is the same age as" (on a set of people)
2. "is a sibling of" (on a set of people)
3. "is a sub-region of" (on a set of geographic areas)
4. "is an ancestor of" (on a set of people)
5. "is taller than or equal to" (on a set of people)

Equivalence Classes

Equivalence

Let $R \subseteq A \times A$ be a relation.

- The relation R is **equivalence** if it is reflexive, symmetric, and transitive
- For $a \in A$, the **equivalence class** $[a]$ is a set of all elements related by an equivalence relation

$$[a] = \{b | \langle a, b \rangle \in R; R \text{ is an equivalence}\}$$

Equivalence Classes Example 1

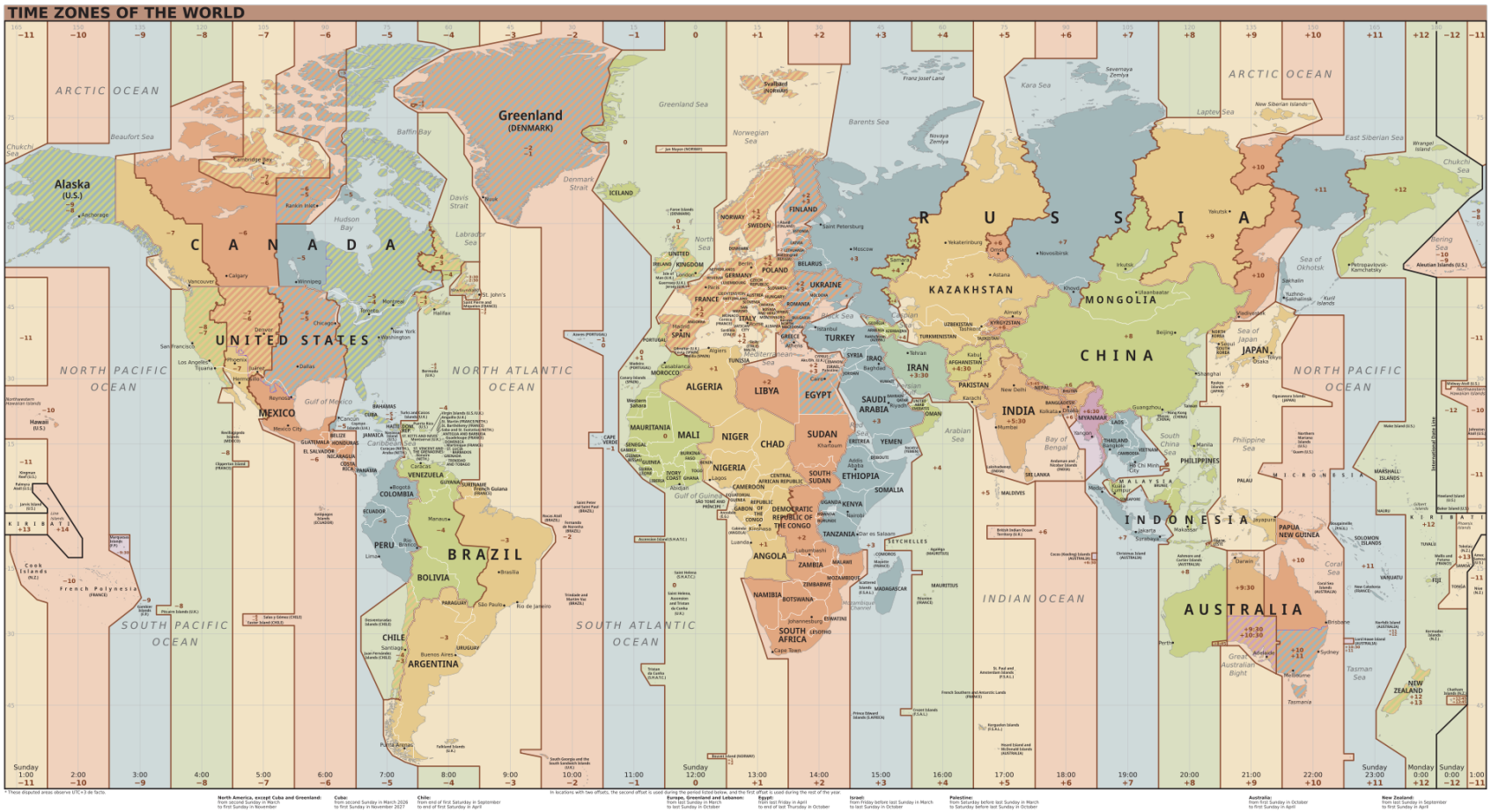
Example (Timezones)

Let A be the set of all cities on Earth. Let us have a relation R which is defined as

"share the same time offset."

Show that R is an equivalence and write down the equivalence classes with respect to R.

Equivalence Classes Example 1



Equivalence Classes Example 2

Example (Parts of Speech)

Let A be the set of words in a vocabulary. Let us have a relation R which is defined as

"can occupy the exact same syntactic slot."

Show that R is an equivalence and write down the equivalence classes with respect to R.

Equivalence Classes Example 2

Example (Parts of Speech)

Equivalence classes with respect to R:

- $[\text{dog}]_R = \{\text{dog, cat, horse, book, ...}\} \rightarrow$ nouns
- $[\text{run}]_R = \{\text{run, sleep, think, sit, ...}\} \rightarrow$ verbs
- $[\text{quick}]_R = \{\text{quick, green, serious, ...}\} \rightarrow$ adjectives

Thanks to Sören!

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